Fractional linear (Möbius) transformations.

Monday, September 18, 2023 9:05 AM

Def. Let $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$ $\left(= \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)$

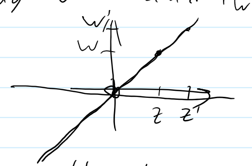
$z \to \dfrac{az+b}{cz+d}$ is called fractional-linear map (or Möbius map)

Rational map of order 1. $\left(\dfrac{az+b}{cz+d}\right)' = \dfrac{ad-bc}{(cz+d)^2}$ - exists when $z \neq -\frac{d}{c}, \infty$

Complex projective line.

Consider $P\mathbb{C} := \mathbb{C}^2 \setminus \{0\} = \left\{ \begin{pmatrix} z \\ w \end{pmatrix}, z, w \in \mathbb{C}, |z|^2 + |w|^2 \neq 0 \right\} / \sim$

$\begin{pmatrix} z \\ w \end{pmatrix} \sim \begin{pmatrix} z' \\ w' \end{pmatrix}$ if $zw' = z'w$ - complex "line" through $\vec{0}$ and $\begin{pmatrix} z \\ w \end{pmatrix}$.



Natural map: $\hat{\mathbb{C}} \leftrightarrow P\mathbb{C}$ $\quad \lambda \neq \infty \Rightarrow \left[\begin{pmatrix} \lambda \\ 1 \end{pmatrix}\right]$

$\lambda = \infty \Rightarrow \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ - an invertible matrix $\mathbb{C}^2 \to \mathbb{C}^2$, preserves lines through origin.

$\left[A\begin{pmatrix} z \\ w \end{pmatrix}\right] = \left[\begin{pmatrix} az+bw \\ cz+dw \end{pmatrix}\right]$ - a map of $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$, given by $\frac{az+b}{cz+d}$. Denoted by T_A.

Some notations: $GL(2,\mathbb{C}) = \{ 2\times 2 \text{ matrices}, \det \neq 0 \}$.

$SL(2,\mathbb{C}) = \{ 2\times 2 \text{ matrices}, \det = 1 \}$.

$A \in GL(2,\mathbb{C}) \Leftrightarrow \frac{1}{\sqrt{\det A}} A \in SL(2,\mathbb{C})$.

Observe: $T_{AB} z = \left[AB\begin{pmatrix} z \\ 1 \end{pmatrix}\right] = \left[A\left(B\begin{pmatrix} z \\ 1 \end{pmatrix}\right)\right] = T_A(T_B z)$. So $T_A^{-1} = T_{A^{-1}}$.

So the map $A \mapsto T_A$ - group homomorphism

$\ker T = \{\pm I\}$ in $SL(2,\mathbb{C})$

$\ker T = \{\lambda I, \lambda \in \mathbb{C} \setminus \{0\}\}$ in $GL(2,\mathbb{C})$

$PSL(2,\mathbb{C}) := SL(2,\mathbb{C}) / \{\pm I\} = GL(2,\mathbb{C}) / \{\lambda I\}$

If you are familiar with groups.

Theorem. Any Möbius map maps lines and circles to lines and circles.

Proof. If $c = 0$, $z \to \frac{a}{d} z + \frac{b}{d}$ $(ad - bc = ad \neq 0 \Rightarrow d \neq 0)$

composition of multiplication by $\frac{a}{d}$ and shift by $\frac{b}{d} \Rightarrow$

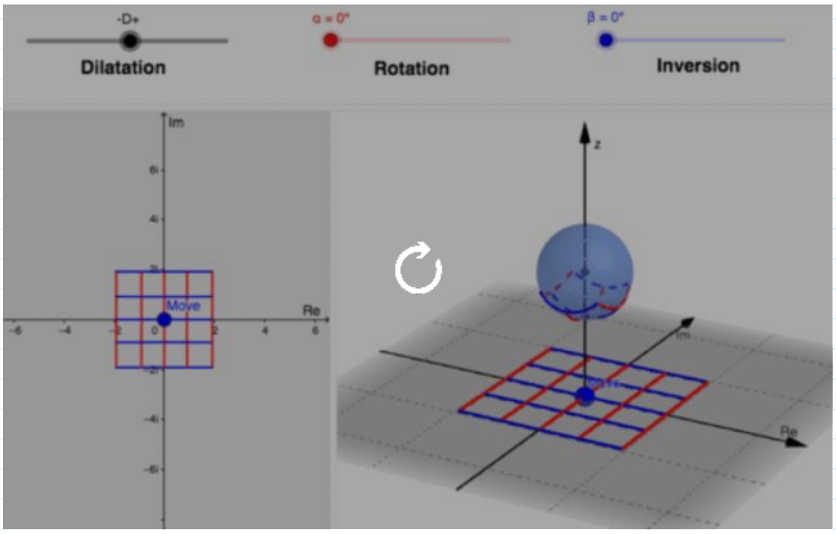
preserves circles and lines.

If $c \neq 0$: $\dfrac{az+b}{cz+d} = \dfrac{a}{c} + \dfrac{bc-ad}{c^2\left(z+\frac{d}{c}\right)}$ – composition of shift by $\frac{d}{c}$, $z \to \frac{1}{z}$ (inversion), multiplication by $\dfrac{bc-ad}{c^2}$, and another shift by $\frac{a}{c}$.

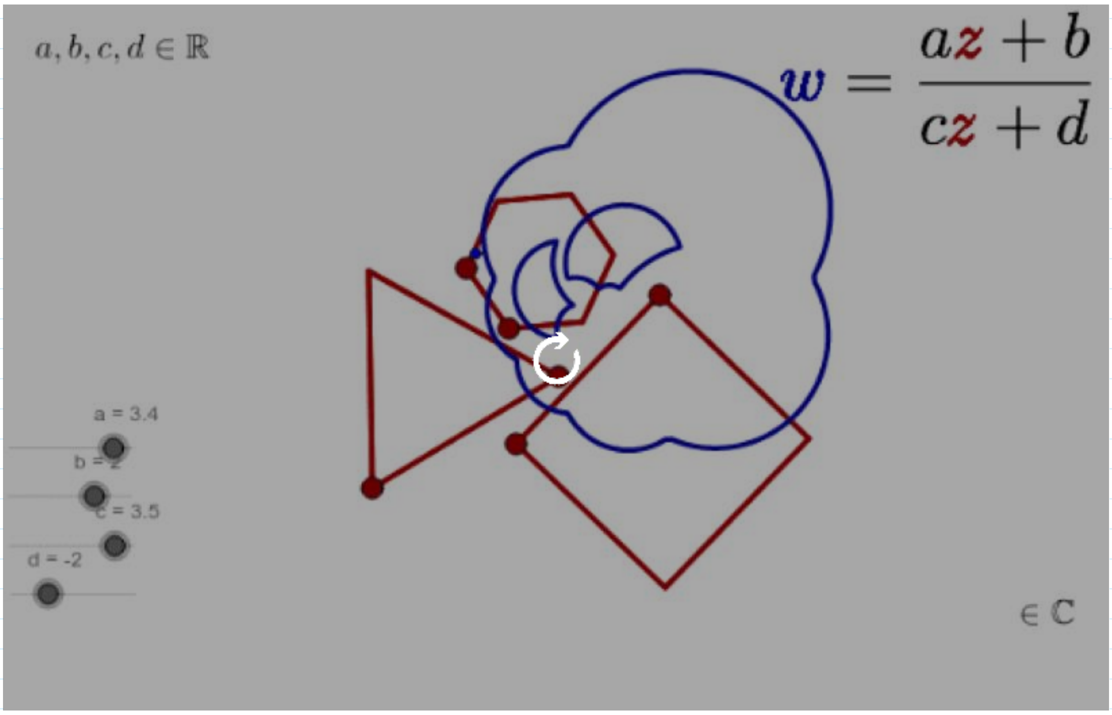
$z \to \frac{1}{z}$ preserves circles and lines, because so does stereographic projection P and $R: (x_1, x_2, x_3) \to (x_1, -x_2, -x_3)$ ($z \to \frac{1}{z} = P \circ R \circ P^{-1}$). So the whole composition also preserves them ■

What we just observed: any Möbius map is a composition of some dilations ($z \to rz$, $r>0$), rotations ($z \to e^{i\theta}z$), translations ($z \to z+a$), and inversions ($z \to \frac{1}{z}$).

[Möbius transformation]



[Mobius transformations]:animation.



$a, b, c, d \in \mathbb{R}$
$w = \frac{az+b}{cz+d}$
a = 3.4
b = 2
c = 3.5
d = -2
$\in \mathbb{C}$